Setup and graphical solution of
Linear Programming Problems [2-variables]

Mathematical Programming Characteristics

Decisions must be made on the levels of a two or more activities. The levels are represented by *decision variables* $X_1 X_2$, etc.

The activities all contribute to some measurable benefit (which we wish to maximize) or to some measurable cost (which we wish to minimize) [This is the *objective* in the problem]. The objective is usually referred to as Z and is given by some function of the decision variables $Z = f(X_1, X_2, \ldots)$.

The levels of the activities are limited by a collection of *constraints* [they require the use of limited resources, or make contributions to meeting required goals, or are subject to rules on their relationships] These are expressed by equations and inequalities involving the decision variables

Assumptions in Mathematical programming models

1. Benefits/costs from different activities can be measured and compared.
2. Benefits/costs from different activities are independent (except as explicitly stated by the constraints)
3. Data are known with certainty (no probability calculations are involved)

Linear Programming Model

1. The function giving the value of the objective is *linear* – that is, of the form $Z = C_1 X_1 + C_2 X_2 + \ldots + C_n X_n$ for some constants $C_1, C_2, \ldots C_n$ (If the variables representing the levels of the activities are $X_1, X_2, \ldots, X_n$)
2. The constraints can be expressed by *linear* equations and inequalities involving only the decision variables.
   linear means: of the form $A_{11}X_1 + A_{12}X_2 + \ldots + A_{1,n}X_n \leq B_1$ or $A_{11}X_1 + A_{12}X_2 + \ldots + A_{1,n}X_n \geq B_1$ or $A_{11}X_1 + A_{12}X_2 + \ldots + A_{1,n}X_n = B_1$)

The Conditions for a problem to fit the Linear Programming Model

1. The parameter values are known with certainty.
2. The objective and constraints exhibit constant returns to scale [Twice as much of any variable quantity has exactly twice the effect, etc.]
3. There are no interactions between the decision variables [A change of +2 in a quantity has the same effect regardless of the values of the other quantities]
4. The decision variables are continuous [fractional values make sense].
   If the decision values must be whole numbers – counts, for example – we have *Integer Linear Programming*– ILP – which adds some complications

Two setup examples
(based on Hillier, Hillier, Lieberman *Introduction to Management Science*)

Example 1
Wyndor Glass company plans to make two new products, 1.) a glass door with aluminum framing and 2.) a double-hung, wood-framed window

They expect each door made & sold to produce a \$300 profit and each window to produce a \$500 profit. They would like to maximize the total profit.

The new products will be manufactured in the companys three existing factories, which have (or can make) time free of other production demands: 4 hrs per week at plant 1, 12 hours per week at plant 2, 18 hours per week at plant 3. Making a door requires 1 hour of work at plant 1 plus 3 hours at plant 3 . Making a window requires 2 hours of work at plant 2 plus 2 hours at plant 3. Costs are already included in the profit figures.

It is helpful to summarize the data on what is available in a table.
In order to standardize, we will use columns for the quantities on which a decision must be made (the decision variables) and rows for the quantities and considerations that put restrictions on the values (the constraints)
We will also put a row for the objective (our measure of success)

Time needed and time available for production

Plant	Doors	windows	Available/Needed
1	1	0	4
2	0	2	12
3	3	2	18
Profit	\$300	\$500	

The decision variables are

D = number of doors to make per week

W = number of windows to make per week

Since production can carry over from week to week, fractional values would make sense

The objective is profit, and the data given show that $Z = 300D + 500W$ (which is linear)

The constraints come from the limitations on available time at the production plants. The time required at plant 1 will be D hours – and this cannot be more than 4 hours . The time required at plant 2 will be $2W$ hours – this cannot be more than 12 hours. The time required at plant 3 will be $3D + 2W$ and cannot be more than 18 hours.

So we want to

Maximize $Z = 300D + 500W$

With constraints

$D \leq 4$

$2W \leq 12$

$3D + 2W \leq 18$

$D \geq 0W \geq 0$ (Nonnegativity constraints – It would not make sense to try to produce a negative number of doors or windows)

Example 2

Profit & Gambit is planning a major advertising campaign for three key products – a prewash stain remover, a liquid laundry detergent, and a powder laundry detergent. These will be advertised in print (including cents-off coupons in magazines and newspapers) and through television ads. The campaign is designed to coordinate all of these, but the company must decide how much advertising to buy in each medium (print vs. television). Each unit of television advertising will cost \$1 million; each unit of print advertising will cost \$2 million.

There are certain minimum goals that have been set up for the campaign: the stain remover should increase its share by at least 3% of its market, the liquid detergent by at least 18% and the powder detergent by at least 4% of the detergent market..

The different media have different efficiencies for different products: It is expected that for each unit of television advertising, the company will gain 3% of market share for the liquid detergent and lose 1% of market share for the powder detergent (because of the competition with the liquid detergent). For each unit of print advertising, the company expects to gain 1% market share for the stain remover, 2% market share for the liquid detergent, and 4% for the powder detergent. The company wants to attain the goals at minimum cost.

Again we can summarize the information in a table (this is typical of linear programming)

Market share requirements and achievement (per unit) for the advertising media:

Product	Television	Print	Required
Stain Remover	0	1%	3%
Liquid Detergent	3%	2%	18%
Powder Detergent	-1%	4%	4%
Cost	\$1 million	\$2 million	

The decision variables are

T = number of units of television advertising

P = number of units of print advertising

The objective is cost

$Z = T + 2P$ (in millions of dollars) and the company wants to minimize this.

The constraints come from the goals that must be met:

Market share gained for the stain remover will be P , and this must be at least 3.

Market share gained for the liquid detergent will be $3T + 2P$ and this must be at lest 18.

Market share gained for the powder detergent will be $-T + 4P$, and this must be a t least 4

Thus we want to

Minimize$Z = T + 2P$
Under the constraints
$P \geq 4$
$3T + 2P \geq 18$
$-T + 4P \geq 4$
$T \geq 0 P \geq 0$ (though these last two constraints turn out to be redundant)

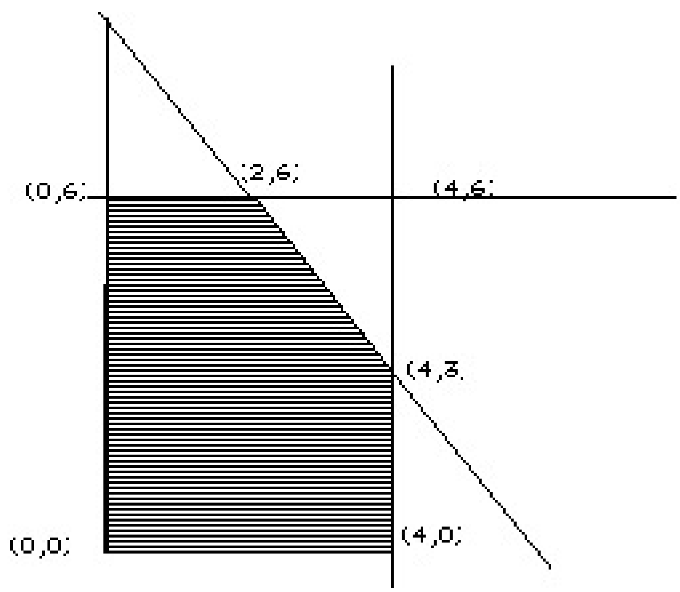
Graphical solution (2 decision variables)

To see how the constraint and objective interact, we will look at the two-variable situation (as in our examples), where we can draw pictures. The same ideas carry over to the more realistic situation with many variables (where we can't draw pictures)
Every possible pair of values for the decision variables is called a "solution" (Think "potential solution" but say "solution")
The constraints set up a set of pairs of values which are "*feasible* solutions" – pairs that are possible (feasible) for the restrictions. We can graph all of these in the "feasible region" of the plane (for two variables we could use 3-dimensional graphing with three variables, but beyond that the ideas work but we cant draw the graphs)

Each restriction (including the Nonnegativity restrictions $X_1 \geq 0, X_2 \geq 0$, etc.) cuts off a region – represented by its graph –made up of points that satisfy the inequality/equation. The overlap of all of these is the "feasible region".

To see the feasible region, we graph the inequalities, in order, and look for the overlap.

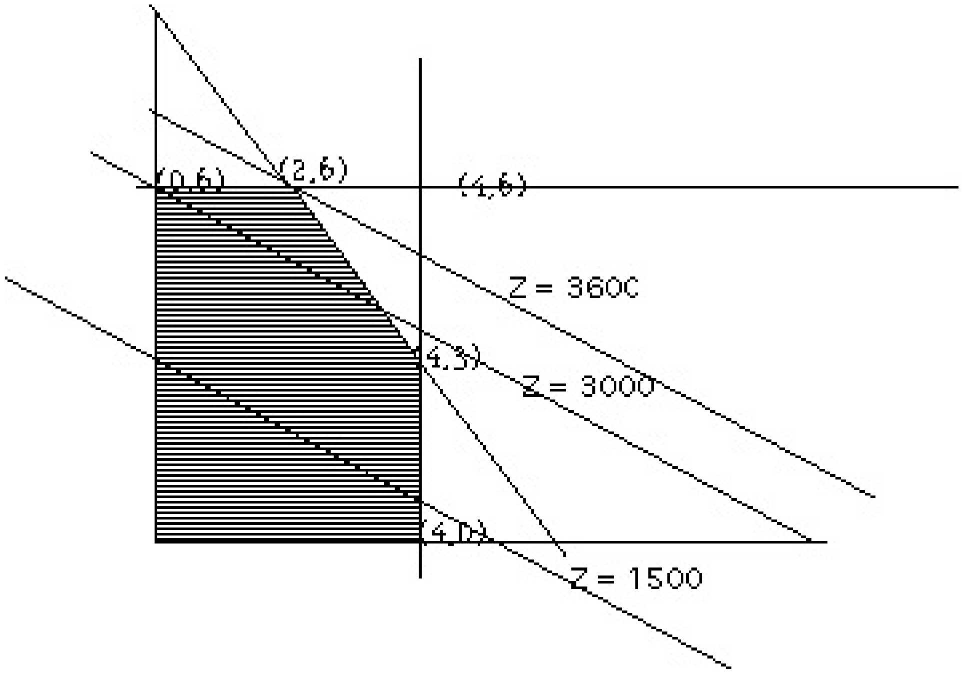
(To graph an inequality: Graph the corresponding equation and determine – by testing a point such as (0,0) – which side of the graph is the graph of the inequality. For linear inequalities the equation will give a line, so finding two points will suffice for graphing)



A particular profit level will be feasible if it can be obtained from some point in the feasible region – for instance, \$1500 is possible with $D = 0$ and $W = 4$, or with $D = 3$ and $W = 1.2$ (remember that fractional values are allowed). In fact, any point on the line $300D+500W = 1500$ will give profit \$1500 – the value is possible if there is a feasible point on the line.

A profit of \$4,000 is not possible – the line $300D + 500W = 4000$ doesnt contain any points of the feasible region.

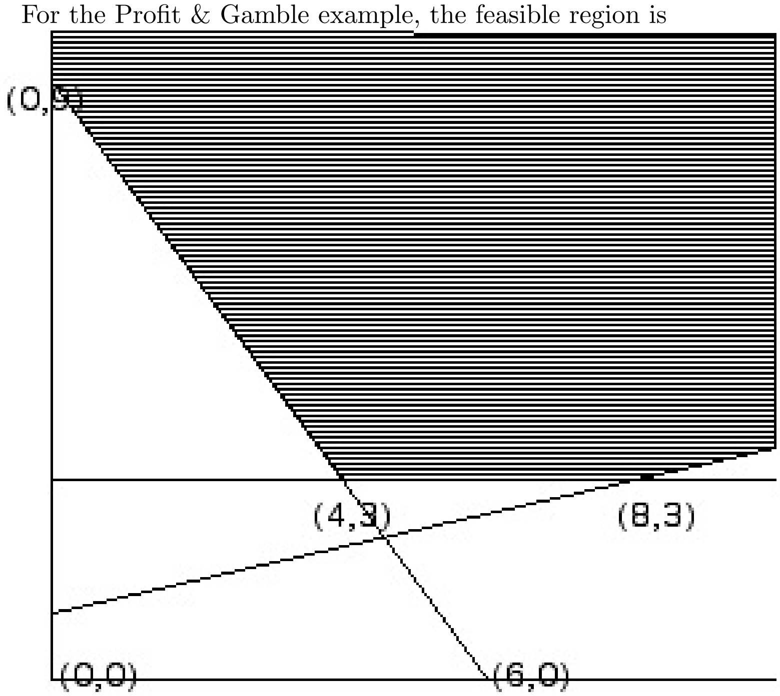
For any profit, the set of points to give that profit is parallel to the line $300D + 500W = 1500$, because for profit Z^* the points are on $300D + 500W = Z^*$ - which has the same slope.
We can find maximum profit by finding the maximum Z^* that gives a line intersecting the feasible region.
Some examples:



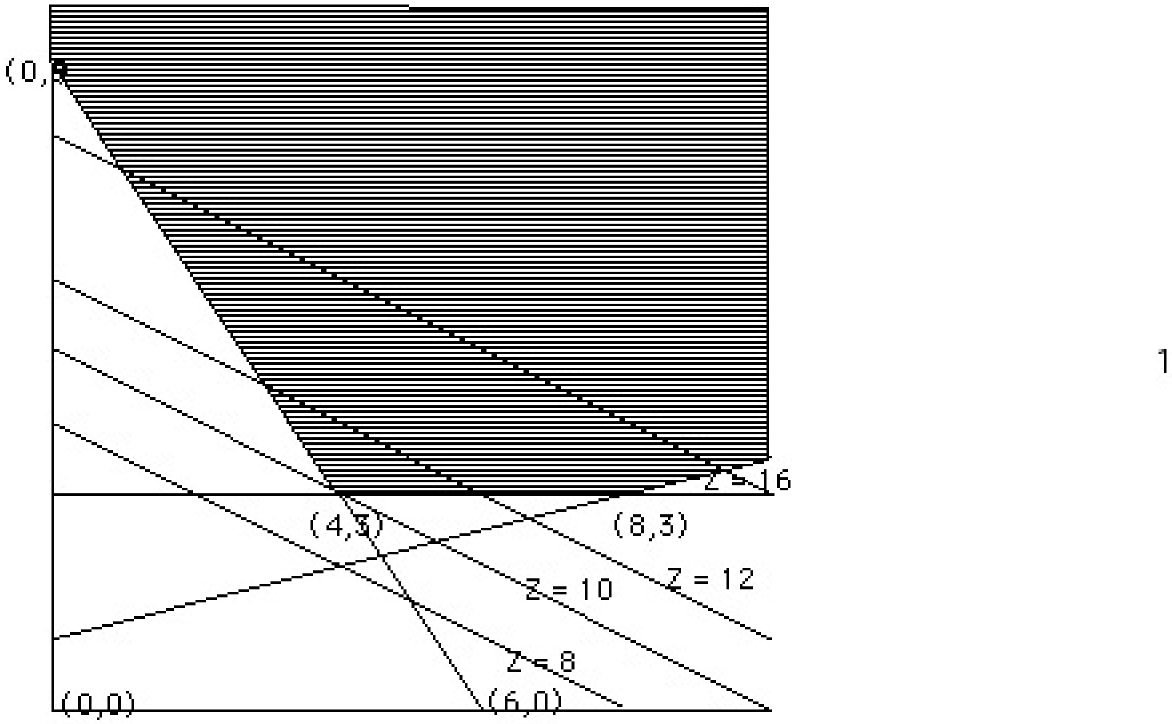
The best profit we can get is \$3600 any attempt at a larger profit (larger value of Z) will give a higher line – that wont hit the feasible region.

If our profits on doors and windows were different – say \$600 for a door and \$300 for a window – wed get a different slope on the profit lines and a different best value but *the line for best profit still would have to hit the feasible region at a corner* (otherwise we could slide the line up & get a better profit).

For the Profit & Gamble example, the feasible region is



Here are some lines for various cost levels:
in order: $Z = 16$, then try smaller – $Z = 12$(still OK), then try $Z = 10$(still OK – hits corner (4,3)) $Z = 8$ (not feasible – no intersection with the feasible region)



The minimum [feasible] cost comes at the corner point (4,3) – any attempt at a lower cost gives a line with no feasible solutions on it..

Thus:

To solve any Linear Programming problem with two decision variables we can use the Graphic method: [see Lawrence & Pasternack p.60]

1. Graph the constraints to find the feasible region
2. Set the objective function equal to an arbitrary value [this gives linear equation] so that the graph [the line for the equation] passes through the feasible region. (Trial and error to get a value that works – or plug in the variable values from an obvious feasible solution)
3. Move the objective function line parallel to itself in the direction of improvement (increase for "Maximize", decrease for "minimize" - may need trial and error to decide what changes) until it touches the last point of the feasible region.
4. Solve for X_1 and X_2 by solving the [system of] two equations that intersect to determine this point.
5. Substitute these values into the objective function to determine its optimal value.

A search method variation on the same idea – involves more computation [solving more systems of equations] but is not as demanding of visual skills [for seeing parallel lines]

1. Graph the constraints [Graph the corresponding equations and see which side is desired] the feasible region is the region in which all overlap
2. Find the corners of the feasible region (the extreme points) by solving the system of two equations that determines each corner [You need the graph to tell which intersections are corners and which are not]
3. Substitute the pair at each corner into the objective function – for a "Maximize" problem, the corner that gives the largest value is the optimal solution. For a "minimize" problem its the corner that gives the smallest value is the optimal solution

(Checking the corners is the heart of this solution method)