Math 438 **ACTIVITY 24:** Constrained Optimization: Kuhn-Tucker Conditions

Why

In general, an optimization problem will involve both equality and inequality constraints. Using and extending the method of Lagrange multipliers, we extend the method of Section 5.1(for non-negativity constraints) to more general inequality constraints and combine them with equality constraints. The resulting set of conditions is called the Kuhn-Tucker conditions, and we consider these now. We also look at one (rather specialized) set of sufficient conditions for optimality.

LEARNING OBJECTIVES

1. Review the methods of finding potential min points for the cases of non-negativity constraints and equality constraints.
2. Understand how our conditions change when we move from equality constraints to inequality constraints.
3. Understand how the Kuhn-Tucker conditions combine the previous methods for dealing with equality constraints and inequality constraints.
4. Be able to use the Kuhn-Tucker conditions to find potential optimal points, and be able to test for optimality.
5. Transfer knowledge from solving unconstrained problems to help in the solution to constrained problems.

CRITERIA

1. Success in completing the exercises.
2. Success in involving all members of the team in the solution
3. Understanding the conditions for the Kuhn-Tucker problem..

RESOURCES

1. Section 5.3 of Strategic Mathematics
2. Typed notes
3. 50 minutes

PLAN

1. Before class: read the text and examples
2. Before class: read the discussion and the model
3. In class: Complete the exercise.

VOCABULARY

Kuhn-Tucker conditions

The following conditions are necessary for a point $\mathbf{X_0}$ to solve the problem:

minimize $f(\mathbf{X})$
subject to $g_k(\mathbf{X}) \geq 0$ for $k = 1, 2, \ldots, K$ [*always* in form "$\geq$"] and $h_j(\mathbf{X}) = 0$ for $j = 1, 2, \ldots, J$.

Define the *Lagrangian Function* $F(\mathbf{X}, \mathbf{U}, \mathbf{V}) = f(\mathbf{X}) - u_k g_k(\mathbf{X}) - v_j h_j(\mathbf{X})$. The necessary conditions for a point $\mathbf{X_0}$ to give a local maximum or local minimum subject to the constraints are called the *Kuhn Tucker Conditions*:

1. $\nabla f(\mathbf{X_0}) - \sum u_k \nabla g_k(\mathbf{X_0}) - \sum v_j \nabla h_j(\mathbf{X_0}) = 0$
2. $g_k(\mathbf{X_0}) \geq 0$ for $k = 1, 2, \ldots, K$
3. $h_j(\mathbf{X_0}) = 0$ for $j = 1, 2, \ldots, J$
4. $u_k g_k(\mathbf{X_0}) = 0$ for $k = 1, 2, \ldots, K$
5. $u_k \geq 0$ for $k = 1, 2, \ldots, K$

Theorem 5.3.2 Let f be convex, the equality constraints all linear and the inequality constraints all concave. If a point $(\mathbf{X_0}, \mathbf{U_0}, \mathbf{V_0})$ satisfies the Kuhn-Tucker conditions, then $\mathbf{X_0}$ is the optimal solution to the problem.

DISCUSSION

The Kuhn-Tucker conditions are *necessary* (but not sufficient) conditions for a point $\mathbf{X_0}$ to be a stationary point for the function, subject to the constraints (a candidate for an optimal point). The theorem gives a set of sufficient (but not necessary) conditions for a point satisfying the first set of conditions to be optimal.

Note the inequality constraints (of the form $g_k \geq 0$) are always converted to "$\geq$" form and are distinguished from the equality constraints (of the form $h_j = 0$). The numbers u_k and v_j are Lagrange multipliers. Conditions 1 and 3 are the "partials with respect to the x_i and h_j are 0" condition included in theorem 5.2.5 (for equality constraints). Conditions 2, 3 say that our optimal point must be a feasible point (satisfying the constraints). Conditions 2, 4, 5 together extend our technique for non-negativity constraints ($x_j \geq 0$) to more general inequalities (directional derivative *along* the boundary must be 0, directional derivative pointing *into* the feasible region must be 0 or positive) and combine them with the requirements for equality constraints.

MODEL

We will consider the problem:

Maximize $3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$
Subject to $2x_1 + x_2 \leq 10, x_1 \geq 0, x_2 \geq 0$

We must first put the problem in proper form, re writing it as:

Minimize $z = -3.6x_1 + 0.4x_1^2 - 1.6x_2 + 0.2x_2^2$
Subject to $10 - 2x_1 - x_2 \geq 0, x_1 \geq 0, x_2 \geq 0$

There are no equality constraints in this problem. We have:

$$F(\mathbf{X}, \mathbf{U}, \mathbf{V}) = -3.6x_1 + 0.4x_1^2 - 1.6x_2 + 0.2x_2^2 - (u_1(10 - 2x_1 - x_2) + u_2x_1 + u_3x_2)$$

the Kuhn-Tucker conditions give us:

1. $\frac{\partial}{\partial x_1}F = -3.6 + 0.8x_1 + 2u_1 - u_2 = 0$ (Condition 1 - first coordinate)
2. $\frac{\partial}{\partial x_2}F = -1.6 + 0.4x_2 + u_1 - u_3 = 0$ (Condition 1, second coordinate)
3. $g_1(x_1, x_2) = 10 - 2x_1 - 2x_2 \geq 0$ (Condition 2, first inequality)
4. $g_2(x_1, x_2) = x_1 \geq 0$ (Condition 2, second inequality)
5. $g_3(x_1, x_2) = x_2 \geq 0$ (Condition 2, third inequality)
6. $u_1g_1 = u_1(10 - x_1 - x_2) = 0$ (Condition 4, first inequality)
7. $u_2g_2 = u_2x_2 = 0$ (Condition 4, second inequality)
8. $u_3g_3 = u_3x_2 = 0$ (Condition 4, third inequality)
9. $u_1 \geq 0$ (Condition 5)
10. $u_2 \geq 0$ (Condition 5)
11. $u_3 \geq 0$ (Condition 5)

We start with Conditions 4 and 5 and consider whether the u_k's can be 0.
If $u_1 = 0, u_2 = 0, u_3 = 0$, then (From 1 and 2) $x_1 = 3.6/.8 = 4.5$ and $x_2 = 1.6/.4 = 4$ but this violates # 3 (because $10 - 2(4.5) - 4 = -3$, which is not at least 0), so the u_k's cannot all be 0.
If $u_1 = 0$ and $u_2 = 0$, then $u_3 > 0$ and #8 says $x_2 = 0$—but then #2 becomes $-1.6 - u_3 = 0$ which is not possible with $u_3 > 0$.
If $u_1 = 0$ and $u_2 > 0$, then #7 says $x_1 = 0$—but then #1 becomes $-3.6 - u_2 = 0$, which is not possible with $u_2 > 0$.
Thus it is not possible for u_1 to be 0. We must have $u_1 > 0$, so $2x_1 + x_2 = 10$ (#6).
If $u_2 = 0$ and $u_3 = 0$, then substituting $10 - 2x_1$ for x_2 (from #3) in (#2), and adding it to (#1), we get $-1.2 + 3u_1 = 0$ or $u_1 = 0.4$. Substituting back into ($1) and ($2) we get $x_1 = 3.5$ and $x_2 = 3$. This satisfies all the conditions and may be the minimum point for z . [Substituting this point into the *original*—to be maximized—function, we would get a maximum 12.9.]
If $u_2 > 0$ and $u_3 = 0$, then (#7) $x_1 = 0$ so $x_2 = 10$ and (#2)$-1.6 + 4 + u_1 = 0$ — which is impossible with

$u_1 > 0$.
If $u_2 > 0$ and $u_3 > 0$ then (#7 & #8) $x_1 = 0 = x_2$—which is impossible because we must have $2x_1 + x_2 = 10$.
If $u_2 = 0$ and $u_3 > 0$ (only remaining case) then $x_2 = 0$ so $x_1 = 5$ and (#1)$-3.6 + 4 + u_1 = 0$, which is impossible with $u_1 > 0$. Thus our only possible point for a minimum for z is $(3.5, 3)$. The Hessian of f is $H = \begin{bmatrix} .8 & 0 \\ 0 & .4 \end{bmatrix}$ which is positive definite everywhere, so f is convex at this point (and at every other point). Since our constraints are all concave (affine functions—so concave), Theorem 5.3.2. says that $(3.5, 3)$ is our optimal points, giving a minimum values-1.29 for z.
Our solution for the original (maximization) problem is $x_1 = 3.5, x_2 = 3$, with value $f = 1.29$

EXERCISE

1. In the model the equality conditions 6, 7, and 8 determine the possible cases we need to look at to exhaust the possible solutions for the Kuhn Tucker conditions. How many possible cases are there in this problem? List them.

2. How many cases did we actually list separately and test in solving the problem in the model?

3. By showing that $u_1 = 0$ and $u_2 > 0$ is not possible, how many of the possible cases [from 1.] did this eliminate (or cover)?

4. Show that: If a problem has only equality constraints and nonnegative variables, the Kuhn Tucker conditions are the same [with different notation] as the conditions given in Theorem 5.2.12. [Hint: In this case all the inequality constraints are of the form $x_k \geq 0$; that is, $g_k(\mathbf{X}) = x_k$ With this information, solve 1st KT condition for u_k's and rewrite the conditions—the v_j (here) will take the place of the λ_j (of theorem 5.2.12) and the h_j of this version are the g_j of 5.2.12.]

5. Set up and solve problem 5.17 on page 186.