Analysis Lab 12

Topic: Uniform Convergence of a Sequence of Functions

Guidelines for Lab Report

For this lab, submit a report according to guidelines given below.

1. Complete Questions 1-3 in Section 2, and write your answers on pages 2-4 of this report guide.
2. Write your answers to Questions 1-3 of Section 3 on pages 5-8 of this report guide.
3. Write your answers to Questions 1-3 of Section 4 on page 9 of this report guide.
4. Complete the Questions for Reflection as assigned by your instructor. Write your response to each question on a separate sheet(s), and attach to the rest of this report.
2 Using Examples to Understand Pointwise Limits

Enter your responses below.

1. \((f_n(x) = x^n)_{n=1}^\infty, [0, 1]\)

(b) Write the first 6 terms of the sequence \((f_n(.5))_{n=1}^\infty\). What is \(\lim_{n \to \infty} f_n(.5)\)?

(c) Does the graph you created in Part (a) support your answer to Part (b)?

(d) Compute the first 6 terms of the sequence \((f_n(.2))_{n=1}^\infty\). What is \(\lim_{n \to \infty} f_n(.2)\)?

(e) Using the graph, what does it appear that \(\lim_{n \to \infty} f_n(.2)\) is?

(f) Write the first 6 terms of the sequence \((f_n(.7))_{n=1}^\infty\). What is \(\lim_{n \to \infty} f_n(.7)\)?

(g) Using the graph, what does it appear that \(\lim_{n \to \infty} f_n(.7)\) is?

(h) For any \(x_0 \neq 1\), what does it appear that \(\lim_{n \to \infty} f_n(x_0)\) is?

(i) What does it appear that \(\lim_{n \to \infty} f_n(1)\) is?

(j) What is the pointwise limit function \(f\)?

(k) Graph the function \(f\).

(l) What can you say about the continuity of each \(f_n\) on the interval \([0,1]\)?

(m) What can you say about the continuity of \(f\) on the interval \([0,1]\)?
2. \( \left( f_n(x) = \frac{x^n}{1 + x^n} \right)_{n=1}^{\infty}, \; [0, 2] \)

(b) Write the first 6 terms of the sequence \( (f_n(0.5))_{n=1}^{\infty} \). What is \( \lim_{n \to \infty} f_n(0.5) \)?

(c) Does the graph you created in Part (a) support your answer to Part (b)?

(d) Write the first 6 terms of the sequence \( (f_n(0.4))_{n=1}^{\infty} \). What is \( \lim_{n \to \infty} f_n(0.4) \)?

(e) Using the graph, what does it appear that \( \lim_{n \to \infty} f_n(0.4) \) is?

(f) Write the first 6 terms of the sequence \( (f_n(0.1))_{n=1}^{\infty} \). What is \( \lim_{n \to \infty} f_n(0.1) \)?

(g) Using the graph, what does it appear that \( \lim_{n \to \infty} f_n(0.1) \) is?

(h) Using the graph, for any \( x_0 \in [0, 1) \), what does it appear that \( \lim_{n \to \infty} f_n(x_0) \) is?

(i) Write the first 6 terms of the sequence \( (f_n(1))_{n=1}^{\infty} \). What does it appear that \( \lim_{n \to \infty} f_n(1) \) is?

(j) Using the graph, what does it appear that \( \lim_{n \to \infty} f_n(1) \) is?

(k) Write the first 6 terms of the sequence \( (f_n(1.2))_{n=1}^{\infty} \). What is \( \lim_{n \to \infty} f_n(1.2) \)?

(l) Using the graph, what does it appear that \( \lim_{n \to \infty} f_n(1.2) \) is?

(m) Using the graph, what does it appear that \( \lim_{n \to \infty} f_n(1.4) \) is?

(n) Using the graph, what does it appear that \( \lim_{n \to \infty} f_n(1.8) \) is?

(o) Using the graph, what does it appear that \( \lim_{n \to \infty} f_n(2) \) is?

(p) Using the graph, for any \( x_0 \in (1, 2] \), what does it appear that \( \lim_{n \to \infty} f_n(x_0) \) is?

(q) What is the pointwise limit \( f \)?

(r) Graph the function \( f \).
(s) What can you say about the continuity of each $f_n$ on the interval $[0,2]$?

(t) What can you say about the continuity of $f$ on the interval $[0,2]$?

3. \( f_n(x) = \lim_{n\to\infty} \left( \frac{x}{1 + nax^2} \right) \), $[0, 1]$

(b) Write the first 20 terms of the sequence \( (f_n(.5))_{n=1}^\infty \). What is $\lim_{n\to\infty} f_n(.5)$?

(c) Does the graph you created in Part (a) support your answer to Part (b)?

(d) Using the graph, what does it appear that $\lim_{n\to\infty} f_n(.2)$ is?

(e) Using the graph, what does it appear that $\lim_{n\to\infty} f_n(1)$ is?

(f) Using the graph, for any $x_0 \in [0, 1]$, what does it appear that $\lim_{n\to\infty} f_n(x_0)$ is?

(g) What is the pointwise limit $f$?

(h) Graph the function $f$.

(i) What can you say about the continuity of each $f_n$ on the interval $[0,1]$?

(j) What can you say about the continuity of $f$ on the interval $[0,1]$?
3 Understanding the Two Types of Convergence

1. \((f_n(x) = x^n)_{n=1}^\infty, [0, 1]\)

(a) \(\epsilon = .5\)

i. Does it appear that \(f_{10}(x) \in (f(x) - \epsilon, f(x) + \epsilon)\) for all \(x \in [0, 1]\)?

   If not, identify those points \(x\) for which \(f_{10}(x) \notin (f(x) - \epsilon, f(x) + \epsilon)\).

ii. Does it appear that \(f_{15}(x) \in (f(x) - \epsilon, f(x) + \epsilon)\) for all \(x \in [0, 1]\)?

   If not, identify those points \(x\) for which \(f_{15}(x) \notin (f(x) - \epsilon, f(x) + \epsilon)\).

iii. Explanation:

(b) \(\epsilon = .2\)

i. Does it appear that \(f_{10}(x) \in (f(x) - \epsilon, f(x) + \epsilon)\) for all \(x \in [0, 1]\)?

   If not, identify those points \(x\) for which \(f_{10}(x) \notin (f(x) - \epsilon, f(x) + \epsilon)\).

ii. Does it appear that \(f_{15}(x) \in (f(x) - \epsilon, f(x) + \epsilon)\) for all \(x \in [0, 1]\)?

   If not, identify those points \(x\) for which \(f_{15}(x) \notin (f(x) - \epsilon, f(x) + \epsilon)\).

iii. Explanation:

(c) \(\epsilon = .1\)

i. Does it appear that \(f_{10}(x) \in (f(x) - \epsilon, f(x) + \epsilon)\) for all \(x \in [0, 1]\)?

   If not, identify those points \(x\) for which \(f_{10}(x) \notin (f(x) - \epsilon, f(x) + \epsilon)\).
ii. Does it appear that $f_{15}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [0, 1]$?

If not, identify those points $x$ for which $f_{15}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

iii. Explanation:

2. \(f_n(x) = \left(\frac{x^n}{1 + x^n}\right)_{n=1}^{\infty}, [0, 2]\)

(a) $\epsilon = .3$

i. Does it appear that $f_{100}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [.95, 1.05]$?

If not, identify those points $x$ for which $f_{100}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

ii. Does it appear that $f_{300}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [.95, 1.05]$?

If not, identify those points $x$ for which $f_{300}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

iii. Explanation:

(b) $\epsilon = .2$

i. Does it appear that $f_{100}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [.95, 1.05]$?

If not, identify those points $x$ for which $f_{100}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

ii. Does it appear that $f_{300}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [.95, 1.05]$?

If not, identify those points $x$ for which $f_{300}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.
iii. Explanation:

(c) $\epsilon = .1$

i. Does it appear that $f_{100}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [.95, 1.05]$?

If not, identify those points $x$ for which $f_{100}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

ii. Does it appear that $f_{300}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [.95, 1.05]$?

If not, identify those points $x$ for which $f_{300}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

iii. Explanation:

3. \( f_n(x) = \frac{x}{1 + n x^2} \) \( n=1 \) to \( \infty \), \([0, 1]\)

(a) $\epsilon = .3$

i. Does it appear that $f_{100}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [0, 1]$?

If not, identify those points $x$ for which $f_{100}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

ii. Does it appear that $f_{300}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [0, 1]$?

If not, identify those points $x$ for which $f_{300}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

iii. Explanation:
(b) $\epsilon = .1$

i. Does it appear that $f_{100}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [0, 1]$?

If not, identify those points $x$ for which $f_{100}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

ii. Does it appear that $f_{300}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [0, 1]$?

If not, identify those points $x$ for which $f_{300}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

iii. Explanation:

(c) $\epsilon = .05$

i. Does it appear that $f_{100}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [0, 1]$?

If not, identify those points $x$ for which $f_{100}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

ii. Does it appear that $f_{300}(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in [0, 1]$?

If not, identify those points $x$ for which $f_{300}(x) \not\in (f(x) - \epsilon, f(x) + \epsilon)$.

iii. Explanation:
4 Critical Thinking Questions

1. Examining the results of the last section, explain in your own words the difference between
the behavior of the sequences from Questions 1 and 2 versus the sequence from Question 3.

2. \( N \)

3. \( (f_n(x) = nxe^{-n^2x})_{n=1}^{\infty}, [0,1] \)
   (b) What is the pointwise limit \( f \)?

(d) \( N \)

(e) What can you say about the continuity of each \( f_n \) on the interval \([0,1]\)?

(f) What can you say about the continuity of \( f \) on the interval \([0,1]\)?