Designing Laboratory Exercises for Upper Division Math Courses

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My goal for this session is to explain to you why and how some colleagues and I have used labs in upper division courses. I intend to be very concrete and hope you will find that I have been. I want to answer your questions and I want to have you spend a good amount of time drafting a lab. Hopefully, I can cover what I think is essential and answer your questions in about 25-30 minutes. Then you will write for about 25 minutes and then we will compare notes. I would like to have each group present some of their conclusions. This is ambitious, but I do want you to have at least a little go at the process instead of merely listening to me.

So here is our plan.

Our Plan for this Session:

- How did we come to design labs?
- How to use labs?
- Topics for labs
- Overview of the structure of a lab
- Nitty Gritty: Questions for a lab
- Other Tips
- Is it worth it?
- Let's try this out

The handout contains contact information, some sample questions, and some sample labs. You can obtain copies of the slides and talk on my website, as well as some other labs. This url is on the handout.

PEOPLE TO CREDIT

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How did we get into the business of designing labs?

- One perspective–my colleague Charlie Peltier

  Charlie came to use labs because he took over a course that had been taught by someone who used only activities or labs in his classes; every class was an activity class. Charlie has embraced the use of labs in that course and has included them in some of his other classes.

  Essentially, Charlie views the labs as a means to improve student learning. Why? He believes, and Kirk and I agree, that students learn more if they talk to each other about the course. Use of the labs forces them to talk and talking forces them to process the material, hence the improvement in the learning. Moreover, students must read to prepare for the labs. This makes them assume more responsibility for their learning. Finally, through the labs and the use of the journal he requires, students have a better qualitative sense of what they are learning. Reflection again improves learning.

- How did Kirk and I come to design the labs we wrote for analysis? Basically, we had the same motivation as Charlie, but we were driven by a particular course – the dreaded analysis.
  - The problems:
    * students had difficulty with the abstractness, in general;
    * students had difficulty understanding basic concepts, in particular, the definitions;
    * students had difficulty with the language;

    * students had difficulty making connections and applying results; and
    * students had difficulty with proofs.
  - Our solution – the labs, for which we had the following goals:
    * have students work with examples to make the abstract concrete,
    * help students develop a working understanding of basic concepts,
    * make the material more accessible–promote success,
    * improve student writing and thinking (which comes first????),
    * help students learn how to learn, and
    * have students experience "doing mathematics" –"thinking like mathematicians".

      Students initially approach analysis the way they approach taking a derivative. They expect the final product, which in the case of analysis is often a proof, to flow from their pen onto the paper. How many of you just sat down at the computer one fine day and typed out your thesis with no preliminary work? Mathematicians use a lot of examples. They use a lot of scrap-paper. We wanted students to experience the acts of getting their hands dirty, of taking baby steps, of looking for patterns, of making conjectures, of making mistakes. These are all essential aspects of doing mathematics.

So having decided to write some labs, how did we use them?

How can the labs be used in class?

For Kirk and myself, this means we ask the students to read the lab ahead of time. The students spend one or two classes working on the lab. They write a report – some of this can be done in class and the rest is completed for the next class. (We give them a report guideline; they do not turn in the answer to every question of the lab.) At the next class, we spend about 10 minutes recapping the lab, making sure that the students recognized the important points.

Charlie also uses the labs to replace class. He may give a short lecture to start a lab, if this is needed.

There are other options. One could assign them as outside activities – done in a group or individually.

Okay, so maybe you are willing to think about trying this. (I will give you student reaction and more plusses later.) How do you decide what to make the subject of the lab?

Topics?

Charlie: By the calendar. He uses the labs once per week in some classes. In Math Programming, the entire course is done in a lab style.

Kirk and myself: What concepts, ideas, or results do students struggle most with? I will give you some points to use to brainstorm for topics.

- KEY Definitions

  You can use a lab to develop a definition or to understand one.

  (The more abstract a definition is, the harder it is for students. The more quantifiers that are involved, the more difficult it is for a a student to grasp.)

  Examples:

  - Analysis – supremum, limit, lim sup and lim inf, continuity, uniform convergence
  - Algebra – group, subgroup, normal subgroup, cosets, order of an element, quotient group, homomorphism, rings, zero divisors
  - Linear Algebra – span, vector space, subspace, linear independence, linear transformation
  - Calculus III – directional derivative, parametric equations, multiple integrals
  - Geometry – compare the definitions of distance in two geometries

- Derive a conjecture. What hypothesis is needed to guarantee a desired result?

  If there is a result that you will use constantly, this is a good one to develop.

  Examples:

  - If I want the set of cosets of a subgroup to be a group, what hypothesis must hold for the subgroup?

- If I want to form an Euler circuit, what condition must the graph satisfy?
- What are some conditions that make a subset a subgroup? Are there certain sets that are always subgroups?
- What can you say about the order of a subgroup in relation to the order of the group?
- When is an element of $\mathbb{Z}_n$ a generator for the group?
- When is there a non-trivial homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}_m$?

- Show the importance of a hypothesis or an axiom. What happens if a given hypothesis or axiom is removed from an important result? Given a hypothesis, what follows from it?

  This helps students understand "if ..., then ..." statements and how to apply such results.

  Examples:

  - Remove "closed" from the hypothesis: If a function is continuous on a closed interval [a,b], then the image is a closed interval.
  - What are the consequences of a sequence being convergent?
  - Explore the importance of each axiom in a geometry.
  - Compare different kinds of geometries – compare the axioms of the two systems.
  - Determine why a given axiom system does not yield the Euclidean geometry.
  - Remove the hypothesis "cyclic" from the statement: If a group G is cyclic and an integer $k/|G|$, then there is an element of order k.
  - What are the consequences of the existence of a homomorphism between two groups?
  - If a group is cyclic, what properties do its subgroups have?

- Prove or formulate your favorite "iff" statement.

  Examples:

  - Linear algebra has the wonderful Invertible Matrix Theorem, which would provide many labs.
  - Let H be a subset of the group G. Then H is a subgroup of G iff ...

  - A finite ring is a field iff it is an integral domain.

- Work through the outline of a proof.

  If a proof is particularly instructive or involves a construction or technique that students will repeatedly use, then it is worth having students work through it.

  Example: Cayley's Theorem: Every group is isomorphic to a group of permutations.

- Reinterpret results in a different setting.
  - Replace the word continuous with differentiable.
  - Replace the word sequence with function.
  - Replace the word "group" with the word "ring".
  - Reinterpret results for measure of lines in the setting of measure of angles.

So in summary, you are looking for key concepts of the course, key results where you want to stress either the importance of each and every hypothesis or the consequence, or equivalent ways to describe an important idea.

So now we have a topic, what will the lab look like?

Structure of a Lab

- Two Notes
  - Use lots of examples and choose them intentionally. You may want to go for some of the classics. You may want to include an example where initially it looks like "Conclusion A" will happen and really "Conclusion B" happens. For example, it may appear, from calculating the first few terms, that a sequence is tending towards infinity when in reality it converges to 0.
  - Written report: Just working through a set of exercises is not enough. Students need to be forced to phrase their ideas in correct mathematical language and they need feedback. Decide whether you want the students to include the answer to every question or just certain questions. We grade for accuracy, clarity, precision, and completeness.

- Charlie's Structure Based on Pacific Crest Process Education The address is on the handout.

  Components of a lab are

  1. Why? Learning Objectives
  2. Criteria
  3. Resources
  4. Plan
  5. Models (optional)
  6. Exercises to be turned in that day
  7. Critical Thinking Questions to be turned in later or completed in the journal

  Show the lab and remind them there are examples in the packet. Note: Charlie's use of roles: captain, spokesperson, recorder, and reflector.

- Our Structure

  1. Introduction
  2. Work with examples – make observations, generate ideas
  3. Think critically about the examples – look for patterns, connect ideas, draw conclusions
  4. Go deeper – additional questions for reflection

  Show the lab and remind them there are examples in the packet.

Let's start writing.

The Nitty Gritty: The Questions

General Remarks:

- Decide what you want to accomplish – that is, why are you writing this lab? Why are the students doing it? Knowing your objectives for a lab helps immensely with the construction of the questions.

  Here is how Charlie phrases this point. Charlie determines what are the key ideas he wants the students to understand and asks "how can I make them do this?" Alternatively, he thinks about how he would present the material – what points he would like to make, what he would use as a primary example. Then he writes questions to lead the students through these points and the example.

- Caveat: Be very careful with the phrasing. Anticipate how a student would answer the question.
- Avoid simple "yes/no" questions or tell students at the start that they must always explain their answers in such cases.
- The majority of questions are pitched so that most of the class will be able to handle them. Nevertheless, ask challenging questions, realizing that even if some answers are weak, it is still worth exposing students to questions that show how mathematicians think. Sometimes one is surprised by which students handle the challenging questions well.
- Some questions can be used to foreshadow future topics.

Different styles of questions – Some examples

- Simple and straightforward: gather info about specific situations or examples

  Examples: Name an upper bound for this set. Name an element in the span of some set of vectors.

- Which of the following statements is true? Ask for a counterexample of the false statement.

  Here you may be trying to find out whether a statement or its converse is true. You can also be trying to help the students determine if they can formulate a true "if ..., then ..." statement. This is a good method to show the "relative strengths" of concepts–which is the stronger or more restrictive concept.

  ### Lab on boundedness of sets

  §1.3.7 In Questions 1-6, you have examined the relationship between boundedness and the existence of a maximum (minimum) and/or supremum (infimum). Consider the following two statements:

  1. If a set of real numbers is bounded above, it has a maximum.
  2. If a set of real numbers is bounded above, it has a supremum.

Only one of these statements is true. Identify the one which is false and give a counterexample. You may use your findings from Section 2. The proof of the true statement is non-trivial. In fact, that statement is sometimes taken as an axiom of the real numbers. Explain why the statement seems reasonable.

- Identify the fallacy in a statement.

  Students find these particularly helpful. If the student can explain why a certain result or definition fails to hold, then he/she must understand the importance of each hypotheses or the logical construction of the definition or theorem. We call these non-examples in my class. In general, I tell students they should construct both examples and non-examples to test their understanding.

  Lab on the Formal Definition of Convergence

  §3.4.2 For each of the following statements, identify the error, and explain how the definition is violated.

  1. Let $(a_n)_{n=1}^{\infty}$ be a sequence, and suppose that $L = 2$. For every $\epsilon > 0$, there exists an N such that $a_n \in (2 - \epsilon, 2 + \epsilon)$ for infinitely many $n > N$. Therefore, $(a_n)_{n=1}^{\infty}$ converges to 2.
  2. Let $(b_n)_{n=1}^{\infty}$ be a sequence, and suppose that $L = 3$. For every $\epsilon > .5$, there exists an N such that $b_n \in (3-\epsilon, 3+\epsilon)$ for all $n > N$. Therefore, $(b_n)_{n=1}^{\infty}$ converges to 3.
  3. Let $(c_n)_{n=1}^{\infty}$ be a sequence, and suppose that $L = -2$. For every N, we can find $\epsilon > 0$ such that $c_n \in (-2 - \epsilon, -2 + \epsilon)$ for all $n > N$. Therefore, $(c_n)_{n=1}^{\infty}$ converges to -2.
  4. Let $(d_n)_{n=1}^{\infty}$ be a sequence, and suppose that $L = 5$. There is an $\epsilon > 0$ such that $d_n \in (5 - \epsilon, 5 + \epsilon)$ for all $n > N$. Therefore, $(d_n)_{n=1}^{\infty}$ converges to 5.

- Focus on quantifiers or investigate how notation is tied to concepts.

  Lab on developing the $\epsilon - \delta$ definition of continuity

  §10.3.2 In those cases in which you were able to find a δ, was the value of δ unique? Could you have selected another value for δ? Could you have selected a maximum value for δ? Could you have selected a minimum value for δ?

§10.3.3 In those cases in which you could not find a δ for a particular ϵ, can you explain what went wrong? In each such case, identify those x in $(x_0 - \delta, x_0 + \delta)$ for which
$f_i(x) \notin (f_i(x_0) - \epsilon, f_i(x_0) + \epsilon)$.

- Questions about objects where you have partial information about the object.

  Here the students must deal with the ideas. They can't hide behind the computations.

  Examples:

- "What if" questions are very helpful. Vary one of the hypotheses and see what happens.

  Examples:

  – Assume students have a standard test to determine when a subset H of a group G is a subgroup. Now ask: what if G finite? Is there a simpler test?

  – In general, one can add the proviso "what if G is finite" to questions about groups.

  – Suppose that students have found a means to determine the generators of $\mathbb{Z}_n$. Now ask, what if n is prime?

- Stretchers

Lab on Conditions Related to Convergence

§7.4.5 Prove that if an increasing, bounded sequence converges, then a non-empty subset of real numbers has a supremum. (Yes, we are asking you to prove that the completeness axiom holds under the given condition.) (Hint: Let S be the bounded set. There exist real numbers a and b such that $S \subset [a, b]$. Cut the interval $[a, b]$ into ten equal subintervals:

$$[\tilde{a}_1 = a, \tilde{b}_1], [\tilde{a}_2, \tilde{b}_2], [\tilde{a}_3, \tilde{b}_3], \ldots, [\tilde{a}_{10}, \tilde{b}_{10} = b].$$

Choose the subinterval which is the last one which contains points of the set; that is, choose the subinterval $[\tilde{a}_k, \tilde{b}_k]$ such that there are points of S in that subinterval, but no points of S in the subinterval $[\tilde{a}_{k+1}, \tilde{b}_{k+1}]$. So the right endpoint of this subinterval is an upper bound of the set; that is, $\tilde{b}_k$ is an upper bound of S. Rename the interval $[\tilde{a}_k, \tilde{b}_k]$ as $[a_1, b_1]$. Repeat the process. Cut the interval $[a_1, b_1]$ into ten equal subintervals. Choose the subinterval which is the last one which contains points of the set; denote that interval as $[a_2, b_2]$. Again the right endpoint of this subinterval is an upper bound of the set. Repeat this process, thus producing a sequence $(a_n)_{n=1}^{\infty}$. Show this sequence converges and relate the limit to the supremum of the set. (Historical note: This is a process due to Karl Weierstrass.)

Other Tips? Concerns

- Don't try to accomplish too much – establish your goals.

- Technology? This is a whole other can of worms. You may find you need no technology. For our labs, sometimes a calculator was sufficient. We also used maple with some labs. To prevent the technology from getting in the way, we wrote Maplets for those labs for which technology was necessary.

Is it worth it? YES!

Advantages:

From the experience of Charlie and myself.

- Class is a positive experience; that is, there is a decrease in frustration level and an increase in success. Students do better in the course when they discuss and write about the mathematics.

  Students gain a better understanding of the concepts. I know they have a better understanding by their success with writing proofs and taking tests.

  Moreover, they can recall examples from the labs when we talk about theorems or concepts in class and apply these results to the topic under discussion.

  How do the labs help with writing and discussing?

  - Obviously the students can't do the labs without discussing the material. The labs provide the students with focused material to discuss.
  - The students are forced to take sloppy intuitive notions and phrase them precisely so that another student or the instructor can understand their point.
  - The students must be accurate and use correct mathematical language in order to answer the questions in the labs and to prepare the reports.
  - Students must write the lab report and are graded on it. This motivates them to be careful and precise.
  - Students learn to read definitions accurately. In particular, they become adept at understanding the role of quantifiers.
- Students' intuition is better, because of their work with examples.
- Students get a solid and working understanding of concepts.

- The process makes the student more responsible for her learning
- Another benefit is that while the instructor is circulating around the room, she learns what difficulties the students are having with the material and can engage in discussion with the students. She can lead them to articulate the ideas correctly on their own. Also the instructor knows what points to address in class.
- The instructor is able to move faster or at least as fast.
- When students join the work force, they will be working on teams. The use of teams now is a minimum risk time to experience working as a member of a team. This experience allows a student to discover her strengths. Charlie chooses the members of the groups deliberately. Each group has a mixture of talent levels. He shuffles the group members at the end of the semester.

Words from students

- First Semester:
  - I felt challenged yet not overwhelmed *all* the time.
  - I probably understood info from labs better than other info in lectures. Some points are reiterated a lot, but this helps in the long run.
  - I thought the labs were definitely helpful in general.
  - I like them because they greatly increase understanding.
  - A lot of work at times, but <u>very</u> helpful. I really gained a lot from them.
  - The labs were hard (but knowledge gained was great)!
  - The lab work was nice because we had to think a lot on our own which is a good thing!
  - The labs were helpful. In groups we could help each other. If one person didn't understand something, there was usually another person who did understand and could help them. I also like the labs because they were more hands-on. We worked with the material to discover the connections and meanings of the material.
  - As for the labs, I have liked the group work. There were times when I have been frustrated by the complexity of the definitions that we are supposed to be developing, but overall, I feel as if they have helped. Examining different cases and different behaviors for sequences has helped so see how the stuff comes together.

  - I have enjoyed the labs because it really make me look at the results and come up with my own analysis.
  - I really enjoyed analysis this semester. I thought the class would be overwhelming with work but it was challenging to the point where it was a bearable frustration. The labs helped me more than I realized at the time. Generating the theorems and concepts with a group allowed me to remember the theorems and concepts better.
  - I think the best part of analysis is the labs. They really helped me me to fully understand the concepts we studied.
- Second semester: "I also liked the labs; I think there should be more of these."

Let's try this out